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On the Teichmüller spaces of Fuchsian groups of Schottky type and the Schwarzian derivatives of univalent functions

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§1. The main result.

Let Γ be a Fuchsian group acting on the upper half plane \mathbb{H} . We denote by $B_2(\Gamma)$ the Banach space of all the holomorphic function φ on \mathbb{H} which satisfies the functional equation $(\varphi \circ \gamma)(\gamma')^2 = \varphi$ for all $\gamma \in \Gamma$, with finite norm $\|\varphi\| = \sup_{z \in \mathbb{H}} |\varphi(z)|(\operatorname{Im} z)^2$. We shall consider the following subsets of $B_2(\Gamma)$:

$$S(\Gamma) = \{\varphi \in B_2(\Gamma) : \exists \text{ univalent function } f \text{ on } \mathbb{H} \text{ with } S_f = \varphi\},$$

$$T(\Gamma) = \{S_f \in S(\Gamma) : f \text{ extends to a } (\Gamma\text{-compatible}) \text{ qc-map of } \hat{\mathbb{C}},$$

where S_f denotes the Schwarzian derivative of f defined as follows: $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$.

It is known that $S(\Gamma)$ is closed and $T(\Gamma)$ is open in $B_2(\Gamma)$. $T(\Gamma)$ is called (the Bers model of) the Teichmüller space of Γ . It is an interesting problem how near $T(\Gamma)$ is to $S(\Gamma)$. For a cofinite Fuchsian group (i.e., finitely generated Fuchsian group of the first kind) Γ , the statement $\overline{T(\Gamma)} = S(\Gamma)$ is equivalent to the Bers conjecture: every B-group is obtained as a boundary group of Teichmüller space. (This conjecture is still now unsolved.)

On the other hand, for any Fuchsian group Γ of the second kind, it is known that $\overline{T(\Gamma)} \subsetneq S(\Gamma)$ (cf. [G2], [Sug]).

But a weaker statement that $T(\Gamma) = \operatorname{Int} S(\Gamma)$ is proved for some cases ([G1: $\Gamma = 1$], [Shiga: cofinite Γ]). The main result of this article is the validity of the above statement for all Fuchsian groups of Schottky type, where a Fuchsian group Γ is called Schottky type in this article, if Γ is a Schottky group simultaneously, in other words, Γ uniformizes a topologically finite Riemann surface of genus g with m holes, where $m \geq 1$. Also, the Schottky type Fuchsian group can be characterized as the finitely generated, purely hyperbolic Fuchsian group of the second kind.

MAIN THEOREM. $\operatorname{Int} S(\Gamma) = T(\Gamma)$ for any Fuchsian group Γ of Schottky type.

§2. Sketch of proof.

Let Γ be a Fuchsian group of Schottky type. Then, the quotient surface $S_0 = \mathbb{H}/\Gamma$ is a topologically finite Riemann surface of genus g with m holes and its double $S = \Omega(\Gamma)/\Gamma$ is a compact Riemann surface of genus $N = 2g + m - 1$, where $\Omega(\Gamma) \subset \hat{\mathbb{C}}$ denotes the region of discontinuity of Γ . Let $\varphi \in \operatorname{Int} S(\Gamma)$ and $F : \mathbb{H} \rightarrow \hat{\mathbb{C}}$ be a holomorphic map such that $S_F = \varphi$. By the Γ -automorphy of

φ , $G = F\Gamma F^{-1}$ is a subgroup of Möb which acts on $D = f(\mathbb{H})$. Since φ is an interior point of $S(\Gamma)$, it turns out that G is purely loxodromic. Since $G(\cong \Gamma)$ is a free group of finite rank, Maskit's characterization theorem tells us that G is also a Schottky group of rank $N = 2g + m - 1$. So, the quotient surface $R = \Omega(G)/G$ is a compact genus N surface. Let $p_0 : \Omega(\Gamma) \rightarrow S$ and $p : \Omega(G) \rightarrow R$ be the natural projections. Set $R_0 = p(D) = D/G$, which is isomorphic to $S_0 = \mathbb{H}/\Gamma$ by the conformal map f induced by $F : \mathbb{H} \rightarrow D$. We shall investigate the way of embedding $R_0 \hookrightarrow R$. Now, the proof of Main Theorem divides into several steps.

STEP 1. ∂R_0 consists of mutually disjoint m simple closed curves.

This step needs a localization of Gehring's method [G1]. In this step, essential is the fact that φ is an interior point of $S(\Gamma)$.

STEP 2. There exists a self-homeomorphism h of R with the following properties:

- (i) $h \circ h = \text{id}_R$,
- (ii) $h|_{\partial R_0} = \text{id}_{\partial R_0}$,
- (iii) $h(R_0) \cap R_0 = \emptyset$,
- (iv) there exists a homeomorphism $H : \Omega(G) \rightarrow \Omega(G)$ such that $p \circ H = h \circ p$ on $\Omega(G)$.

This step is covered by rather algebraic arguments. For example, the following lemma is utilized.

LEMMA (GENERAL PROPERTY OF THE NORMAL COVERINGS).

Suppose that $p : (\Omega, z_0) \rightarrow (R, a_0)$ is a normal covering between (connected) pointed manifolds. Let R_0 be a subdomain of R such that $a_0 \in R_0$ and $\iota : R_0 \rightarrow R$ denote the inclusion map. Then ι naturally induces the homomorphism $\iota_* : \pi_1(R_0, a_0) \rightarrow \pi_1(R, a_0)$. Let $\lambda : \pi(R, a_0) \rightarrow G$ be the lifting homomorphism with respect to z_0 , where G is a covering transformation group of $p : \Omega \rightarrow R$. Namely, $g = \lambda[\alpha]$ for $g \in G$ and $[\alpha] \in \pi_1(R, a_0)$ iff the final point of the lift $\tilde{\alpha}$ of α with initial point z_0 coincides with $g(z_0)$. Then, the followings hold.

- (i) Each component of $p^{-1}(R_0)$ is simply connected $\iff \lambda \circ \iota_*$ is injective.
- (ii) $p^{-1}(R_0)$ is connected $\iff \lambda \circ \iota_*$ is surjective.

In particular, if $p^{-1}(R_0)$ is a simply connected domain, then $\iota_* : \pi_1(R_0, a_0) \rightarrow \pi_1(R, a_0)$ is an embedding and $\pi_1(R, a_0) = \ker \lambda \rtimes \pi_1(R_0, a_0)$ (semi-direct product).

First of all, we can naturally extend f to a homeomorphism $f : \overline{S_0} \rightarrow \overline{R_0}$ by Step 1. Further, by use of Step 2, we can extend f to a homeomorphism $\tilde{f} : S \rightarrow R$

in the following way.

$$\tilde{f} = \begin{cases} f & \text{on } \overline{S_0}, \\ h \circ f \circ j & \text{on } S \setminus \overline{S_0}, \end{cases}$$

where j denotes the involution map $S \rightarrow S$ induced by conjugation $J(z) = \bar{z}$. By construction, \tilde{f} can be lifted, that is, there exists a homeomorphism $\tilde{F} : \Omega(\Gamma) \rightarrow \Omega(G)$ such that $p \circ \tilde{F} = \tilde{f} \circ p_0$. By purely topological arguments, it turns out that \tilde{F} can be naturally extended to a homeomorphism $\tilde{F} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. In particular, it is known that D is an image of \mathbb{H} under the self-homeomorphism \tilde{F} of $\hat{\mathbb{C}}$, so D is a Jordan domain.

STEP 3. ∂R_0 is a disjoint union of quasi-analytic curves.

Here, the “quasi-analytic curve” means the quasiconformal image of a circle. For the proof of Step 3, we need just more delicate arguments than in Step 1. By the way, one can prove the following

PROPOSITION. *Let S and R be compact Riemann surfaces and $S_0 \subset S, R_0 \subset R$ be subdomains with quasi-analytic boundaries. Suppose that $\tilde{f} : S \rightarrow R$ is an orientation preserving homeomorphism such that $\tilde{f}(S_0) = R_0$ and the restriction map $\tilde{f}|_{S_0} : S_0 \rightarrow R_0$ is quasiconformal. Then, there exists a quasiconformal map $\tilde{f}_1 : S \rightarrow R$ which is homotopic to \tilde{f} and $\tilde{f}_1 = \tilde{f}$ on R_0 .*

By virtue of this proposition, we can choose a quasiconformal $\tilde{f} : S \rightarrow R$ as the extension of f . Then, a topological extension $\tilde{F} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ of a lift of \tilde{f} is quasiconformal on $\Omega(\Gamma)$, so $\tilde{F} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a quasiconformal self-homeomorphism since $\Lambda(\Gamma) = \hat{\mathbb{C}} \setminus \Omega(\Gamma) \subset \hat{\mathbb{R}}$ is a quasiconformally removable set. Therefore $D = \tilde{F}(\mathbb{H})$ is a quasi-disk, the proof is completed.

REFERENCES

- [G1]. F.W.Gehring, *Univalent functions and the universal Teichmüller space*, Comment. Math. Helv. **52** (1977), 561–572.
- [G2]. F.W.Gehring, *Spirals and the universal Teichmüller space*, Acta Math. **141** (1978), 99–113.
- [Shiga]. H.Shiga, *Characterization of quasi-disks and Teichmüller spaces*, Tôhoku Math. J. **37** (1985), 541–552.
- [Sug]. T.Sugawa, *On the Bers conjecture for Fuchsian groups of the second kind*, J. Math. Kyoto Univ. **32** (1992), 45–52.